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MICROLOCAL HOLMGREN'S THEOREM FOR A CLASS OF HYPO-ANALYTIC STRUCTURES

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ABSTRACT. A microlocal version of Holmgren's Theorem is proved for a certain class of the hypo-analytic structures of Baouendi, Chang, and Treves.

1. INTRODUCTION

In [4] Sjöstrand gave a simpler proof of a result of Schapira [3] concerning a microlocal version of Holmgren's theorem for real analytic data. Inspired by [4], in this paper we will extend Schapira's result to a certain class of hypo-analytic structures. The paper is organized as follows: In §2 we discuss the Cauchy-Kovalevska theorem for maximal hypo-analytic structures. In §3 we introduce a class of hypo-analytic structures which we call real hypo-analytic, give a statement of the main theorem of this article, and derive two corollaries. A lemma is included in the same section and is used in the proof of the main theorem which appears in §4.

2. CAUCHY-KOVALEVSKA FOR HYPO-ANALYTIC STRUCTURES

We are interested in the hypo-analytic structures introduced by Baouendi, Chang, and Treves in [1]. We briefly recall the relevant concepts here.

Let Ω be a smooth manifold of dimension m. A hypo-analytic structure of maximal dimension on Ω is the data of an open covering $\{U_\alpha\}$ of Ω and for each index α, of m C^∞ functions $Z_\alpha^1, \dots, Z_\alpha^m$ satisfying the following two conditions:

(1) $dZ_\alpha^1, \dots, dZ_\alpha^m$ are linearly independent at each point of U_α;

(2) if $U_\alpha \cap U_\beta \neq \emptyset$, there are open neighborhoods O_α of $Z_\alpha(U_\alpha \cap U_\beta)$ and O_β of $Z_\beta(U_\alpha \cap U_\beta)$ and a holomorphic map F_β^α of O_α onto O_β such that $Z_\beta = F_\beta^\alpha \circ Z_\alpha$ on $U_\alpha \cap U_\beta$.

We will use the notation $Z_\alpha = (Z_\alpha^1, \dots, Z_\alpha^m) : U_\alpha \mapsto C^m$. A distribution h defined in an open neighborhood of a point p_0 of Ω is hypo-analytic at p_0 if there is a chart (U_α, Z_α) of the above type whose domain contains p_0 and a

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holomorphic function $\tilde{h}$ defined on an open neighborhood of $Z_\alpha(p_0)$ in C^m such that $h = \tilde{h} \circ Z_\alpha$ in a neighborhood of p_0. By a hypo-analytic local chart we mean an $m+1$-tuple $(U, Z^1, \dots, Z^m)$ [abbreviated (U, Z)] consisting of an open subset U of Ω and of m hypo-analytic functions whose differentials are linearly independent at every point of U.

In [2] we introduced hypo-analytic differential operators which by definition map hypo-analytic functions to hypo-analytic functions. A linear differential operator P on Ω is hypo-analytic if and only if for every hypo-analytic local chart $(U, Z^1, \dots, Z^m)$, U sufficiently small, and vector fields $M_1, \dots, M_m$ satisfying $M_j Z^k = \delta_j^k$ we have: $P = \sum_{|\alpha| \le n} a_\alpha(x) M^\alpha$, where each a_α is a hypo-analytic function on U. Let p be an arbitrary point of Ω. The differentials of the germs of hypo-analytic functions at p make up a complex vector subspace of the complex cotangent space $CT_p^*\Omega$. This subspace, which we denote by T_p', has dimension $= m$. Condition (2) in the definition of hypo-analytic structures implies that the subspace T_p' makes up a smooth vector subbundle T' of the complex cotangent bundle $CT^*\Omega$. T' will be referred to as the structure bundle.

We now introduce the concept of hypo-analytic submanifolds. By a submanifold of Ω we mean a subset of Ω equipped with a C^∞ structure such that the natural injection into Ω is a C^∞ map with injective differential. Let M be a submanifold of Ω. We shall denote by π_M the natural map $T^*\Omega_{|M} \mapsto T^*M$ and by π_M^C the analogous map of the complex cotangent bundles. In general, $T_M' = \pi_M^C(T')$ is not a vector bundle.

Definition 2.1. A submanifold M of Ω is called a hypo-analytic submanifold if it is equipped with a hypo-analytic structure whose structure bundle is identical to T_M' and which has the following property: Given any hypo-analytic function f on an open set $\Omega' \subset \Omega$ which intersects M, the restriction of f to $M \cap \Omega'$ is hypo-analytic.

Simple examples show that the second property in the above definition is not redundant.

Proposition 2.1. *Suppose Σ is a hypo-analytic submanifold of Ω whose structure bundle has dimension $m - k$. Then each point $q \in \Sigma$ is contained in a hypo-analytic chart $(U; Z^1, \dots, Z^m)$ of Ω with $Z^{m-k+1}, \dots, Z^m$ all vanishing on $U \cap \Sigma$.*

Proof. Let $q \in \Sigma$ and $(U; W^1, \dots, W^m)$ be a hypo-analytic chart for Ω around q. Since the differentials $dW^1, \dots, dW^m$ span CT^*U, without loss of generality we may assume that $\pi_\Sigma^C(dW^1), \dots, \pi_\Sigma^C(dW^{m-k})$ span $CT^*(U \cap \Sigma)$.

Moreover, $(U \cap \Sigma, W_{|\Sigma}^1, \dots, W_{|\Sigma}^{m-k})$ is a hypo-analytic chart in Σ since Σ is a hypo-analytic submanifold of Ω.

Now $W^{m-k+1}, \ldots, W^m$ all restrict to hypo-analytic functions in Σ. Therefore, there are holomorphic functions $H_1, \ldots, H_k$ such that $W^{m-k+j}(x) = H_j(W^1(x), \ldots, W^{m-k}(x))$ for each $x \in \Sigma \cap U$ and $1 \leq j \leq k$. Here the set U may have to be contracted. For $x \in U$, let

$$Z^j(x) = W^j(x), \qquad 1 \leq j \leq m-k,$$

and

$$Z^l(x) = W^{m-k+l}(x) - H_l(W^1(x), \ldots, W^{m-k}(x))$$

when $m-k \leq l \leq m$.

Then $(U; Z^1, \ldots, Z^m)$ is a hypo-analytic chart on Ω satisfying the properties in the proposition.

Remark 2.1. If Σ is a hypo-analytic submanifold of Ω, then the dimension of Σ is the same as the dimension of its structure bundle.

Suppose now P is a hypo-analytic differential operator on Ω. We would like to introduce the concept of noncharacteristic hypersurfaces. Let Σ be a hypo-analytic hypersurface of Ω. By Proposition 2.1, Σ is locally given by $H(x) = 0$, where H is hypo-analytic and $dH \neq 0$. If $(U; Z^1, \ldots, Z^m)$ is a hypo-analytic chart for Ω near a central point $q \in \Sigma$, then P can be written as $P = \sum_{|\alpha| \leq k} a_\alpha(Z(x))M^\alpha$ and $H(x) = \tilde{H}(Z(x))$ for some holomorphic functions a_α and $\tilde{H}$ in a neighborhood of $Z(q)$ in C^m. We push everything by the map Z into C^m near $Z(q)$ and write $P^Z(z, \frac{\partial}{\partial z}) = \sum_{|\alpha| \leq k} a_\alpha(z)(\frac{\partial}{\partial z})^\alpha$ and $\Sigma^Z = \{z \in C^m : \tilde{H}(z) = 0\}$.

Since $dH \neq 0$, Σ^Z is a complex submanifold of C^m of complex codimension 1 passing through $Z(q)$.

If $(V; W^1, \ldots, W^m)$ is another hypo-analytic chart about q, let G be a biholomorphism near $Z(q)$ in C^m such that $(W^1, \ldots, W^m) = G(Z^1, \ldots, Z^m)$. Then $P_k^W(w, \frac{\partial}{\partial w})$ and Σ^W are the expressions of $P_k^Z(z, \frac{\partial}{\partial z})$ and Σ^Z in the coordinates $w^1, \ldots, w^m$ of C^m. Hence, in particular, Σ^Z is noncharacteristic with respect to P^Z if and only if Σ^W is noncharacteristic with respect to P^W.

This observation justifies the following definition in which we use the same notations as above.

Definition 2.2. We say Σ is noncharacteristic with respect to P at a point $q \in \Sigma$ if Σ^Z is noncharacteristic with respect to $P^Z(z, \frac{\partial}{\partial z})$ at $Z(q)$ for some hypo-analytic chart $(U; Z^1, \ldots, Z^m)$ about q.

We can now formulate a Cauchy-Kovalevska theorem for a hypo-analytic differential operator and hypo-analytic Cauchy data on a noncharacteristic hypo-analytic hypersurface.

Suppose now P is a hypo-analytic differential operator and Σ is a noncharacteristic hypo-analytic hypersurface with respect to P at the point $q \in \Sigma$. Let the order of P near $q = k$. Suppose L is a hypo-analytic vector field not belonging to $CT\Sigma$ at the point q (and hence near q). Then we have:

Theorem 2.1. *There is an open neighborhood Ω' of q in Ω such that to every hypo-analytic function f in Ω' and to every set of k hypo-analytic functions $u_0, \dots, u_{k-1}$ on $\Sigma \cap \Omega'$, there is a unique hypo-analytic function u in Ω' such that*

$$Pu = f \quad in\ \Omega',$$

and for every $j = 0, \dots, k-1$, $L^j u = u_j$ in $\Sigma \cap \Omega'$.

Proof. By Proposition 2.1, $q \in \Sigma$ is contained in a hypo-analytic chart $(U; Z^1, \dots, Z^m)$ of Ω with Z^m vanishing on $U \cap \Sigma$. Let $M_1, \dots, M_m$ be the vector fields in U satisfying $M_j Z^k = \delta_j^k$. Then in the chart (U, Z), we may write $P = \sum_{|\alpha| \le k} a_\alpha(x) M^\alpha$ and $L = \sum_j c_j(x) M_j$, where the coefficients are all hypo-analytic. The condition $L \notin CT\Sigma$ near q is equivalent to $c_m(x) \neq 0$ for x near q.

Let $\tilde{u}_j$, $\tilde{f}$, $\tilde{a}_\alpha$, and $\tilde{c}_j$ be the holomorphic functions defined near $Z(q) \in C^m$ such that $u_j(x) = \tilde{u}_j(Z(x))$ etc.

Set

$$P^Z\left(z, \frac{\partial}{\partial z}\right) = \sum_{|\alpha| \le k} a_\alpha(z) \left(\frac{\partial}{\partial z}\right)^\alpha,$$

$$L^Z = \sum_{j=1}^m \tilde{c}_j(z) \frac{\partial}{\partial z_j} \quad \text{and} \quad \Sigma^Z = \{z \in C^m : z_m = 0\}.$$

The assumptions on Σ and L imply that Σ^Z is noncharacteristic for P^Z and that $\tilde{c}_m(z) \neq 0$ for z near $Z(q)$. Therefore the existence part of Theorem 2.1 follows from the existence part of the holomorphic version of the Cauchy-Kovalevska theorem applied to the problem

$$P^Z \tilde{u} = \tilde{f} \quad \text{near } Z(q) \text{ in } C^m$$

and for $0 \le j \le k-1$,

$$(L^Z)^j \tilde{u} = \tilde{u}_j \quad \text{near } Z(q) \text{ in } \Sigma^Z \text{ (see [7]).}$$

We just set $u(x) = \tilde{u}(Z(x))$ and observe that $M_j u(x) = \frac{\partial \tilde{u}}{\partial z_j}(Z(x))$ for each $j = 1, \dots, m$. To see the uniqueness, suppose u' is another solution and set $v = u - u'$. Then

$$Pv = 0 \quad \text{in } \Omega' \qquad \text{and} \qquad L^j v = 0 \quad \text{in } \Sigma \cap \Omega'$$

and v is hypo-analytic. Since $M_1, \dots, M_{m-1}$ all belong to $CT\Sigma$ and $v = 0$ on Σ, it follows that $M_1 v = \cdots = M_{m-1} v = 0$ on Σ (near q). Now $L = \sum_{j=1}^m c_j(x) M_j$ with $c_m(x) \neq 0$ and $Lv = 0$ on Σ. Therefore $M_m v = 0$ on Σ. Moreover, from $L^j v = 0$ for $0 \le j \le k-1$, we deduce that $M^\alpha v = 0$ for $|\alpha| \le k-1$ on Σ. Next, since the coefficient of M_m^k in $P = \sum_{|\alpha| \le k} a_\alpha(x) M^\alpha$

is nonzero, it follows that on Σ, $M^\alpha v = 0$ for $|\alpha| \le k$. Finally, applying the vector fields M_j to the equation $Pv = 0$, we see that $M^\alpha v = 0$ on Σ for all indices α. Now let $\tilde{v}$ be the holomorphic function near $Z(q)$ in C^m satisfying $v(x) = \tilde{v}(Z(x))$.

We write the power series of v around $Z(q)$ as

$$\tilde{v}(z) = \sum a_\alpha (z - Z(q))^\alpha, \qquad \text{where } a_\alpha = \frac{1}{\alpha!}\left(\frac{\partial}{\partial z}\right)^\alpha \tilde{v}(Z(q)).$$

But then

$$\left(\frac{\partial}{\partial z}\right)^\alpha \tilde{v}(Z(q)) = (M^\alpha v)(q) = 0 \qquad \forall \alpha.$$

Therefore, $\tilde{v} \equiv 0$ near $Z(q)$. Hence $v \equiv 0$ in Ω'.

3. Real hypo-analytic structures and statement of the main result

We will continue to look at a maximal hypo-analytic structure on Ω. We noted that a hypersurface Σ is hypo-analytic if and only if Σ is the zero set of a hypo-analytic function f with nonzero differential. We now strengthen this condition and introduce the following:

Definition 3.1. Σ is said to be a real hypo-analytic hypersurface if every point $p \in \Sigma$ has a neighborhood U_p in Ω, a hypo-analytic function h of a nonzero differential defined on U_p, and $\varepsilon > 0$ such that:

(1) $\Sigma \cap U_p = \{x \in U_p : h(x) = 0\}$.

(2) For $c \in C$, $|c| < \varepsilon$, the set $\Sigma_c = \{x \in U_p : h(x) = c\}$ is either $\varnothing$ or a hypersurface.

(3) $\bigcup \Sigma_c$ is a neighborhood in U_p of p; $|c| < \varepsilon$.

We note that near each point of Σ, the above definition gives a local foliation of Ω by means of hypo-analytic hypersurfaces.

Example 1. Suppose Ω is a real analytic structure. The real analytic structure can be viewed as a hypo-analytic structure and in this case, any real analytic hypersurface is real hypo-analytic.

Example 2. Consider a hypo-analytic local chart (U, Z) around 0 in a maximal hypo-analytic structure on R^m. Suppose $Z_j = x_j + \sqrt{-1}\,\phi_j(x)$, $j = 1, \dots, m-1$, and $Z_m = x_m + \sqrt{-1}\,\phi_m(x_m)$, where $\phi = (\phi_1, \dots, \phi_m)$ is real-valued, with zero differential at 0, and $\phi(0) = 0$.

Assume that U is small enough so that the mapping $Z = (Z_1, \dots, Z_m) : U \to C^m$ is a diffeomorphism of U onto $Z(U)$. Then $\Sigma = \{x \in U : x_m = 0\}$ is a real hypo-analytic hypersurface. In this case, the defining function can be taken to be Z_m.

Lemma 3.2 will show that Example 2 is a typical example.

The proof of the main theorem will use two equivalent formulations of microlocal hypo-analyticity that were developed in [1]. We briefly recall them here.

Sato's Microlocalization. We consider a hypo-analytic local chart (U, Z) of the maximal structure Ω.

In the sequel Γ is a nonempty, acute, and open cone in $R^m\backslash\{0\}$. For A an open subset of U and $\delta > 0$, let

$$N_\delta(A, \Gamma) = \{Z(x) + \sqrt{-1}\, Z_x(x)v : x \in A, v \in \Gamma, |v| < \delta\}.$$

Let $B_\delta(A, \Gamma)$ denote the space of holomorphic functions on $N_\delta(A, \Gamma)$ of tempered growth. More precisely, a holomorphic function f with domain $N_\delta(A, \Gamma)$ is in $B_\delta(A, \Gamma)$ if it satisfies the condition: to every compact subset K of $N_\delta(A, \Gamma)$ there are an integer $k \geq 0$ and a constant $c > 0$ such that $|f(z)| \leq c(\text{dist}[z, Z(A)])^{-k}$ for all z in K.

In [1] it was shown that if A is sufficiently small and $f \in B_\delta(A, \Gamma)$, then for every $\psi \in C_c^\infty(A)$,

$$\lim_{t \to +0} \int_A f(Z(x) + \sqrt{-1}\, Z_x(x)tv)\, \psi(x)\, dZ(x)$$

exists and is independent of $v \in \Gamma$. Let bf denote the limit distribution.

Definition 3.2. Let $u \in D'(U)$ and $(x, \xi) \in U \times R_m\backslash\{0\}$. We say that u is microlocally hypo-analytic at (x, ξ) if there are an open neighborhood $A \subseteq U$ of x, $\delta > 0$ and a finite collection of nonempty acute open cones Γ_k in $R_m\backslash\{0\}$ $(k = 1, \ldots, r)$ satisfying $\langle v, \xi\rangle < 0$ for every v in each Γ_k and such that the the following hold:

for each k there is $f_k \in B_\delta(A, \Gamma_k)$ such that in A,

$$u = bf_1 + \cdots + bf_r.$$

The above definition of microlocal hypo-analyticity in the cotangent space does not depend on the choice of the chart (U, Z) (see [1]).

Definition 3.3. Let $u \in D'(\Omega)$. The hypo-analytic wavefront set of the distribution u is denoted by $\text{WF}_{\text{ha}}u$ and is defined as

$$\text{WF}_{\text{ha}}u = \{(x, \xi) \in T^*\Omega : u \text{ is not hypo-analytic at } (x, \xi)\}.$$

The FBI Transform. We continue to work in a chart (U, Z) of the maximal structure Ω. Assume that $Z = (Z_1, \ldots, Z_m) : U \to C^m$ is a diffeomorphism of U onto $Z(U)$ and that U is the domain of local coordinates x_j $(1 \leq j \leq m)$ all vanishing at a "central point" which will be denoted by 0. We will suppose $Z(0) = 0$ and by substituting $Z_x(0)^{-1}Z(x)$ for $Z(x)$ if necessary, we may assume that

$$Z_x(0) = \text{the identity matrix.}$$

Let u be a compactly supported distribution in U. We shall refer to

$$F(u, z, \zeta) = \int_y \exp(\sqrt{-1}\, \zeta \cdot (z - Z(y)) - \langle\zeta\rangle(z - Z(y))^2) u(y) dZ(y)$$

as the Fourier-Bros-Iagolnitzer (in short, FBI) transform of u. Here $z \in C^m$, $\zeta \in C_m$ with $|\mathrm{Im}\,\zeta| < |\mathrm{Re}\,\zeta|$, and

$$\langle\zeta\rangle^2 = \zeta_1^2 + \cdots + \zeta_m^2.$$

In [1], the authors established the following FBI transform criterion for hypo-analyticity. We will state it here in a form that will be of convenience to us.

Theorem 3.1. *The following two properties of a compactly supported distribution are equivalent*:

(i) *u is microlocally hypo-analytic at $(0, \xi^0) \in T^*U\backslash\{0\}$.*

(ii) *There is an open neighborhood V of 0 in C^m, a conic open neighborhood $\mathscr{C}_0$ of ξ^0 in C_m, and constants $c, r > 0$ such that $|F(u, z, \zeta)| \le c\exp(-r|\zeta|)$ for all z in V and for all ζ in $\mathscr{C}_0$.*

We are now ready to state the main theorem of this paper.

Theorem 3.2. *Let P be a hypo-analytic differential operator and Σ a real hypo-analytic hypersurface which is noncharacteristic for P. Assume $u \in D'(\Omega)$ such that Pu is hypo-analytic. Suppose $\sigma \in T^*\Omega\,|_{\Sigma}$ for which the hypo-analytic Cauchy data of u are microlocally hypo-analytic at $\pi_\Sigma(\sigma)$. Then $\sigma \notin \mathrm{WF}_{\mathrm{ha}}u$.*

Remark 3.2. The proof will actually show that it is sufficient to have Pu microlocally hypo-analytic at σ.

From Theorem 3.2 we deduce the following consequences. Σ and P will be as in Theorem 3.2.

Corollary 3.1. *Suppose Pu is hypo-analytic at $q \in \Sigma$ and the hypo-analytic Cauchy data of u is also hypo-analytic at q. Then u is hypo-analytic at q.*

Proof. Since the hypo-analytic Cauchy data is hypo-analytic at q, it is microlocally hypo-analytic in every direction in $T_q^*\Sigma/\{0\}$. (See [1] for a proof.) Therefore, by Theorem 3.2, u is microlocally hypo-analytic in every direction in $T_q^*\Omega$. Hence by [1], u is hypo-analytic at q.

Corollary 3.2. *Suppose $Pu = 0$ and the hypo-analytic Cauchy data of u on Σ is 0. Then $u \equiv 0$.*

Proof. By Corollary 3.1, u is hypo-analytic. But then by the uniqueness part of Theorem 2.1, $u \equiv 0$.

The following lemmas will be used in the proof of Theorem 3.2.

Lemma 3.1. *Let P be a hypo-analytic differential operator and $\sigma \notin$ Char P. If $u \in \mathscr{D}'(\Omega)$ for which $\sigma \notin \mathrm{WF}_{\mathrm{ha}}Pu$, then $\sigma \notin \mathrm{WF}_{\mathrm{ha}}u$.*

Proof. We reason in a chart (U, Z) around 0 where we assume that $Z(0) = 0$, $dZ(0) = \mathrm{Id}$, $\sigma = (0, \xi^0) \in T^*U$, and U is the domain of local coordinates

x_j $(1 \le j \le m)$. We can then take $\Re Z_j$ as new coordinates in which $Z(x) = x + \sqrt{-1}\phi(x)$, $\phi(0) = 0$, $d\phi(0) = 0$ and $\phi = (\phi_1, \ldots, \phi_m)$ is real-valued. Moreover, the functions Z_j may be selected so that all the derivatives of order 2 of the ϕ_j vanish at 0. Indeed, if this is not already so it suffices to replace each Z_j by

$$Z_j - \frac{\sqrt{-1}}{2} \sum_{k=1}^{m} \sum_{l=1}^{m} \frac{\partial^2 \phi_j}{\partial x_k \partial x_l}(0) Z_k Z_l .$$

Let M_j $(1 \le j \le m)$ be the vector fields satisfying $M_j Z_k = \delta_j^k$. To prove the lemma, we will use the FBI transform. First we note that for any $f \in C^1(U)$,

$$\langle df, M_k \rangle = M_k f = \sum_j \langle (M_j f) dZ_j, M_k \rangle \qquad \forall k.$$

It follows that

$$df = \sum_{j=1}^{m} (M_j f) dZ_j.$$

Therefore, if g or h has compact support in U, by Stokes' theorem we have

$$\begin{aligned} 0 &= \int_{\partial U} hg\, dZ_1 \wedge \cdots \wedge \widehat{dZ_j} \wedge \cdots \wedge dZ_m \\ &= (-1)^{j-1} \left[\int_U [(M_j h) g + h(M_j g)] dZ_1 \wedge \cdots \wedge dZ_m \right]. \end{aligned}$$

Hence

$$\int_U (M_j h) g\, dZ_1 \wedge \cdots \wedge dZ_m = - \int_U h(M_j g)\, dZ_1 \wedge \cdots \wedge dZ_m. \tag{3.1}$$

If U is sufficiently small, in the chart (U, Z) we may write

$$P = \sum_{|\alpha| \le k} a_\alpha(x) M^\alpha ,$$

where each a_α is hypo-analytic on U.

Since $\sigma = (0, \xi^0) \notin \mathrm{WF}_{\mathrm{ha}} Pu$, Theorem 3.1 tells us that

$$\begin{aligned} &F(Pu, z, \zeta) \\ &\quad = \int_U \exp(\sqrt{-1}\zeta \cdot (z - Z(y)) - \langle \zeta \rangle (z - Z(y))^2) \sum_{|\alpha| \le k} a_\alpha(y) M^\alpha u(y)\, dZ(y) \end{aligned}$$

has an exponential decay for z near 0 and ζ in a complex conic neighborhood of ξ^0.

Since $y \mapsto \exp(\sqrt{-1}\zeta \cdot (z - Z(y)) - \langle \zeta \rangle (z - Z(y))^2)$ is hypo-analytic, for each $j = 1, \ldots, m$,

$$M_j(\exp h(z, \zeta, y)) = [-\sqrt{-1}\zeta_j + 2\langle \zeta \rangle (z_j - Z_j(y))] \exp(h(z, \zeta, y)),$$

where

$$h(z, \zeta, y) = \sqrt{-1}\zeta \cdot (z - Z(y)) - \langle\zeta\rangle(z - Z(y))^2.$$

This observation together with the integration by parts formula (3.1) imply the existence of a hypo-analytic amplitude $Q(z, \zeta, y)$ elliptic at σ such that

$$F(Pu, z, \zeta) = \int_U \exp(\sqrt{-1}\zeta \cdot (z - Z(y)) - \langle\zeta\rangle(z - Z(y))^2)Q(z, \zeta, y)u(y)\, dZ.$$

By the results of [5], we conclude that $\sigma \notin \mathrm{WF}_{\mathrm{ha}}u$.

Lemma 3.2. *Suppose Σ is a real hypo-analytic hypersurface of Ω. Then each point $p \in \Sigma$ is contained in a hypo-analytic chart $(U, Z_1, \dots, Z_m)$, where U is the domain of local coordinates $(U, x_1, \dots, x_m)$ in which*

$$Z_j = x_j + \sqrt{-1}\phi_j(x) \quad \textit{for } 1 \le j < m$$

and $Z_m = x_m + \sqrt{-1}\Psi(x_m)$, where

$$(\phi_1, \dots, \phi_{m-1}, \Psi) \textit{ is real-valued and } \Sigma \cap U = \{x \in U : x_m = 0\}.$$

Proof. By Proposition 2.1, there is a chart (U, Z) of Ω near p such that

$$\Sigma \cap U = \{x : Z_m(x) = 0\}.$$

Since $d(Z_1|_\Sigma), \dots, d(Z_{m-1}|_\Sigma)$ are linearly independent, by making linear substitutions if necessary, we may assume that $d(\Re Z_1|_\Sigma), \dots, d(\Re Z_{m-1}|_\Sigma)$ are independent.

We may then take $\Re Z_1, \dots, \Re Z_{m-1}$ as coordinates on Σ. By multiplying Z_m by $\sqrt{-1}$ if necessary, we may also assume that $\Re Z_1, \dots, \Re Z_m$ are coordinates in U (all this locally near p)

Then

$$Z_j = x_j + \sqrt{-1}\phi_j, \quad Z_m = x_m + \sqrt{-1}\Psi(x), \qquad 1 \le j < m,$$

and since $Z_m|_{\Sigma\cap U} = 0$, we have

$$\Sigma \cap U = \{x \in U : x_m = 0\}.$$

Next let h be the defining function of Σ near p satisfying the conditions of Definition 3.1. Write $h(x) = \tilde{h}(Z(x))$, where $\tilde{h}$ is holomorphic.

Since $h|_\Sigma = \tilde{h}(Z_1, \dots, Z_{m-1}, 0)|_\Sigma = 0$ and the image of Σ under $(Z_1, \dots, Z_{m-1})$ is a totally real manifold of maximal dimension in C^{m-1}, it follows that

$$h(x) = \tilde{h}(Z_1(x), \dots, Z_m(x)) = \tilde{h}(Z_m(x)).$$

Now since $dh \neq 0$, $\tilde{h}$ is invertible. Hence, for any constant $c \in C$,

$$h(x) = c \quad \text{iff} \quad Z_m(x) = \tilde{h}^{-1}(c).$$

It now follows from Definition 3.1 that $Z_m = x_m + \sqrt{-1}\Psi(x_m)$.

4. Proof of Theorem 3.2

Lemma 3.2 permits us to reason in a local hypo-analytic chart (U, Z), where U is also the domain of local coordinates $(U, x_1, \ldots, x_m)$ centered at 0 with $Z_j = x_j + \sqrt{-1}\phi_j(x)$, $1 \le j < m$, $Z_m = x_m + \sqrt{-1}\Psi(x_m)$, Σ is given by $x_m = 0$ and $\sigma = (0, \xi_0)$.

We may also assume that $Z(0) = 0$, $dZ(0) = \mathrm{Id}$, $\phi''(0) = 0$, and $\Psi''(0) = 0$.

Let M_j $(1 \le j \le m)$ be the vector fields satisfying $M_j Z_k = \delta_j^k$. If $p \in \Sigma$ and $1 \le j < m$, then $(M_j)_p \in CT_p\Sigma$. Moreover, after multiplication by a nonvanishing hypo-analytic function, P will have the form

$$P = M_m^n + \sum_{|\alpha| \le n,\, \alpha_m < n} a_\alpha(x) M^\alpha,$$

where the a_α are all hypo-analytic functions. Since Pu is hypo-analytic, it follows that u is a C^∞ function of x_m valued in the space of distributions in the variable $x' = (x_1, \ldots, x_{m-1})$ (see [8]). In particular, the trace of u on Σ is well defined.

We may therefore restate the theorem as:

Suppose Pu is hypo-analytic and $(0', \xi') \in T^*_{0'}\Sigma$ such that $(0', \xi'_0) \notin \mathrm{WF}_{\mathrm{ha}}(M_m^j u(x', 0))$ for $0 \le j < n$. Then $(0, (\xi'_0, \xi_n)) \notin \mathrm{WF}_{\mathrm{ha}} u$.

Since the statement is purely local, we may assume that the support of u is contained in a set of the form

$$\{x' : |x'| \le T/2\} \times (-T, T) \quad \text{and} \quad \{(x', 0) : |x'| \le T\} \subseteq \Sigma.$$

For $t \in (-T, T)$, let $\Sigma_t = \Sigma \times \{t\}$ and $\Omega_t = \{(x', x_m) : |x'| < T/2, 0 \le x_m < t \text{ or } t < x_m \le 0\}$.

We observe that for any j, k, and l,

$$M_j(M_k Z_l) = 0 = M_k(M_j Z_l).$$

Since the differentials $dZ_1, \ldots, dZ_m$ span CT^*U, it follows that the vector fields M_j commute pairwise. This observation together with the integration by parts formula of §3 and the fact that for each t and $j < m$, $M_j \in CT\Sigma_t$ yield:

$$\begin{aligned} &\int_{\Omega_t} (Pu) w \, dZ_1 \wedge \cdots \wedge dZ_m - \int_{\Omega_t} u({}^tPw)\, dZ_1 \wedge \cdots \wedge dZ_m \\ &= \sum_{j+k \le n-1} \int_{\Sigma_t} (M_m^j u)(B_{jk}(x, M') M_m^k w)\, dZ_1 \wedge \cdots \wedge dZ_{m-1} \\ &\quad - \sum_{j+k \le n-1} \int_{\Sigma_0} (M_m^j u)(B_{jk}(x, M') M_m^k w)\, dZ_1 \wedge \cdots \wedge dZ_{m-1}, \end{aligned} \tag{4.1}$$

where the B_{jk} are hypo-analytic differential operators in $M_1, \ldots, M_{m-1}$ of order $n - 1 - j - k$.

For $\alpha = (z_0', \xi') \in C^{m-1} \times (R_{m-1}\backslash\{0\})$ and $\tau \in C$ satisfying $1 < |\tau| < C_0$, $|\Im\tau| < \varepsilon\Re\tau$ (ε and C_0 to be determined later), set

$$V_{\alpha,\tau}(z') = \exp(\sqrt{-1}(z_0' - z')\cdot\xi' - \tau|\xi'|(z_0' - z')^2).$$

Since ${}^tP$ is a hypo-analytic differential operator, let

$${}^tP = \sum_{|\alpha|\le n} c_\alpha(x) M^\alpha,$$

where each $c_\alpha(x) = \tilde{c}_\alpha(Z(x))$ for holomorphic $\tilde{c}_\alpha$. Set

$${}^tP\left(z, \frac{\partial}{\partial z}\right) = \sum_{|\alpha|\le n} \tilde{c}_\alpha(z)\left(\frac{\partial}{\partial z}\right)^\alpha.$$

Let $\tilde{\Sigma}_t = \{(z', t) \in C^{m-1} \times \{t\} : |z'| \le T\}$.

The Cauchy-Kovalevska theorem tells us that there is $t_0 > 0$ such that if $t \in [-t_0, t_0]$ we can find a solution $\tilde{w}(z) = \tilde{w}_{\alpha,\tau,t}(z)$ in a neighborhood of $\{(z', x_m) \in C^{m-1} \times R : |z'| \le T, |x_m| < t_0\}$ of the problem

$$(4.2)\qquad \begin{gathered} {}^tP\left(z, \frac{\partial}{\partial z}\right)\tilde{w} = 0, \qquad \tilde{w}|_{\tilde{\Sigma}_t} = \cdots = \left(\frac{\partial}{\partial z_m}\right)^{n-2}\tilde{w}|_{\tilde{\Sigma}_t} = 0 \\ \left(\frac{\partial}{\partial z_m}\right)^{n-1}\tilde{w}|_{\tilde{\Sigma}_t} = V_{\alpha,\tau}. \end{gathered}$$

The solution $\tilde{w} = \tilde{w}_{\alpha,\tau,t}$ can be estimated in terms of the Cauchy data on $\tilde{\Sigma}_t$. Indeed, the Ovcyannikov method (see [6]) implies

$$\exists c > 0 \text{ independent of } t, \tau, \alpha \quad \text{such that}$$

$$|\tilde{w}_{\alpha,\tau,t}(w', z_m)| \le c \sum_{|\beta'|\le n} \sup_{|z'-w'|\le c|z_m - t|} |\partial_{z'}^{\beta'} V_{\alpha,\tau}(z')|.$$

For $|\beta'| \le n$ we have

(4.4)

$$|\partial_{z'}^{\beta'} V_{\alpha,\tau}(z')| \le c_1(|1 + |\xi'|)^n \exp(\langle\Im(z' - z_0'), \xi'\rangle - |\xi'|[\Re\tau\{(\Re z' - \Re z_0')^2\} - 2\Im\tau\Re(z' - z_0')\cdot\Im(z' - z_0')]).$$

We are going to be interested in z', z_0', where $\Im z'$ is small compared to $\Re z'$ and z_0' is close enough to $0'$. This consideration together with a sufficiently small choice of ε in the definition of τ imply for $|\beta'| \le n$

$$(4.5)\quad |\partial_{z'}^{\beta'} V_{\alpha,\tau}(z')| \le c_1(| + |\xi'|)^n \cdot \exp\left(\langle\Im(z' - z_0'), \xi'\rangle - \frac{\Re\tau}{2}|\xi'|[(\Re z' - \Re z_0')^2 - (\Im z' - \Im z_0')^2]\right).$$

Application of (4.5) to (4.3) yields

$$|\tilde{w}_{\alpha,\tau,t}(z', x_m + i\Psi(x_m))| \\ \le c_1(| + |\xi'|)^n \exp\left(\langle \Im(z' - z_0'), \xi'\rangle - \frac{\Re\tau}{2}|\xi'| \times [(\Re z' - \Re z_0')^2 - (\Im z' - \Im z_0')^2] + c|\xi'||x_m - t|\right).$$

Let $w_{\alpha,\tau,t}(x) = \tilde{w}_{\alpha,\tau,t}(Z(x))$. For $\alpha = (z_0', \xi')$ in a sufficiently small conic neighborhood of $(0', \xi_0')$ and with $w = w_{\alpha,\tau,t}$ we wi.. estimate the term

$$\int_{\Omega_t} (Pu)w\, dZ_1 \wedge \cdots \wedge dZ_m \quad \text{in (4.1).}$$

(4.2) tells us that $w = w_{\alpha,\tau,t}$ solves

$$(4.2') \qquad {}^tP(x, M)w = 0, \qquad w|_{\Sigma_t} = \cdots = M_m^{n-2}w|_{\Sigma_t} = 0,$$

and

$$M_m^{n-1}w(x', t) = V_{\alpha,\tau}(Z(x', t)).$$

Since Pu and $w = w_{\alpha,\tau,t}$ are hypo-analytic, we can deform the integration contour from Ω_t to the image of Ω_t under the map

$$(x', x_m) \mapsto \theta(x', x_m) = Z(x', x_m) - \sqrt{-1}\left(d\chi(x')\frac{\xi'}{|\xi'|}, 0\right),$$

where $\chi(x')$ is a cutoff function $\equiv 1$ near $\Re z_0'$ and d is chosen so that we stay inside the domain of hypo-analyticity.

Along this contour, (4.6) gives the following estimate on $w = w_{\alpha,\tau,t}$:

$$(4.7) \quad |w| \le c_1(| + |\xi'|)^n \\ \times e^{(-d\chi(x')|\xi'| + \langle\phi'(x), \xi'\rangle - \frac{\Re\tau}{2}|\xi'|[(x' - \Re z_0')^2(\phi'(x) - d\chi(x')\frac{\xi'}{|\xi'|} - \Im z_0')^2] + c|\xi'||x_m - t|)}.$$

(Here $\phi' = (\phi_1, \ldots, \phi_{m-1})$.)

By using the term $(x' - \Re z_0')^2$ when x' is away from $\Re z_0'$ and the term $d\chi(x')|\xi'|$ when x' is near $\Re z_0'$, we see that w is exponentially decaying along this contour. The latter may require shrinking of the interval $[-t_0, t_0]$ to a smaller interval which we will still call $[-t_0, t_0]$.

It follows that we can find a sufficiently small $t > 0$ and a sufficiently large $c_2 > 0$ such that

$$(4.8) \qquad \left|\int_{\Omega_t} (Pu)w_{\alpha,\tau,t}\, dZ_1 \wedge \cdots \wedge dZ_m\right| \le c_2 \exp\left(-\frac{|\xi'|}{c_2}\right)$$

for $|t| \le t_0$ and $\alpha = (z_0', \xi')$ in a small conic neighborhood of $(0', \xi_0')$.

Since $w = w_{\alpha,\tau,t}$ solves (4.2′), formula (4.1) reduces to

(4.9)
$$i(-1)^{n+1}\int_{\Omega_t}(Pu)w\,dZ_1\wedge\cdots\wedge dZ_m$$
$$=\int_{|x'|\le T} e^{(\sqrt{-1}\langle z_0'-Z'(x',t),\xi'\rangle-\tau|\xi'|(z_0'-Z'(x',t))^2)}$$
$$\times u(x',t)\,dZ_1\wedge\cdots\wedge dZ_{m-1}(x',t)$$
$$+\,i(-1)^n\sum_{j+k\le n-1}\int_{\Sigma_0}(M_m^j u)(B_{jk}(x,M')M_m^k w)\,dZ_1\wedge\cdots\wedge dZ_{m-1}(x',0).$$

We consider now the integrals over $\Sigma_0 = \Sigma$. Fix j and $k \ni j+k\le n-1$. Since by assumption $(0',\xi')\notin \mathrm{WF}_{\mathrm{ha}}(M_m^j u|_{\Sigma_0})$, without loss of generality we may assume

$$M_m^j u|_{\Sigma_0} = \lim_{s\downarrow 0} f_j(Z'(x',0)+\sqrt{-1}sZ'_{x'}(x',0)v)$$

$(Z' = (Z_1,\ldots,Z_{m-1}))$ for some tempered holomorphic function f_j, and v is in a cone $\Gamma_j\subseteq R^{m-1}$ satisfying

$$\langle v,\xi_0'\rangle < 0.$$

Hence, in the integral over Σ_0, we may deform a contour to $Z(x',0)+\sqrt{-1}s\chi(x')Z_x(x',0)v$, where s is chosen sufficiently small and $\chi(x')$ is selected as before.

Estimates analogous to (4.6) are also valid for the derivatives $\{M_m^k w\}_k$. Such estimates and the new contour for each j yields, after enlarging c_2 if necessary,

$$\left|\int_{\Sigma_0}(M_m^j u)B_{jk}(x,M')M_m^k w\,dZ_1\wedge\cdots\wedge dZ_{m-1}\right|\le c_2\exp\left(-\frac{|\xi'|}{c_2}\right) \tag{4.10}$$

for $t\in[-t_0,t_0]$ and $\alpha=(z_0',\xi_0')$ in a small conic neighborhood of $(0',\xi_0')$.

It follows that (after modifying t_0 and c_2)

(4.11)
$$\left|\int_{|x'|\le T}u(x',t)\exp(\sqrt{-1}\langle z_0'-Z'(x',t),\xi'\rangle-\tau|\xi'|(Z'(x',t)-z_0')^2)dZ'\right|$$
$$\le c_2\exp\left(-\frac{|\xi'|}{c_2}\right)$$

for $t\in[-t_0,t_0]$ and $\alpha=(z_0',\xi')$ in a small conic neighborhood of $(0',\xi_0')$.

Let $I(t,\tau,z_0',\xi')=$ the integral (without the absolute value) in (4.11). Suppose (4.11) holds in a cone $\Gamma'\subseteq R^{m-1}$ containing ξ_0'.

Let $z_0 = (z'_0, z^m_0) \in C^{m-1} \times C$ and $\xi = (\xi', \xi_m) \in R_{m-1} \times R$. In order to examine $\mathrm{WF}_{\mathrm{ha}} u$ at $(0, (\xi'_0, \xi_m))$, we have to estimate the FBI:

$$F(z_0, \xi) = \int_{|t| \le t_0} \int_{|x'| \le T} \exp(\sqrt{-1}\langle z_0 - Z(x', t), \xi\rangle \\ - |\xi|(z_0 - Z(x', t))^2) u(x', t)\, dZ.$$

But since Z_m depends only on t, we get

$$F(z_0, \xi) = \int_{|t| \le t_0} \exp((z^m_0 - Z_m(t))\xi_m - |\xi|(z^m_0 - Z_m)^2) \\ \times I(t, |\xi|/|\xi'|, z'_0, \xi')\, dZ_m(t).$$

We now select C_0 as follows. Since $(0, \xi^0) \notin \mathrm{Char}\, P$, by Lemma 3.1 $\exists$ a constant $C_0 > 1$ such that if $|\xi| \ge C_0|\xi'|$, then $F(z_0, \xi)$ decays exponentially for z_0 near 0 in C^m.

Let $\Gamma = \Gamma' \times R$. Pick $\xi = (\xi', \xi_m) \in \Gamma$. To finish the proof, we consider two cases:

Case (i). $|\xi| \ge C_0|\xi'|$. This was just taken care of.

Case (ii). $|\xi| \le C_0|\xi'|$. Then $|\xi| = (|\xi|/|\xi'|)|\xi'| = \tau|\xi'|$ with $1 < \tau < C_0$.

Hence (4.11) and (4.12) guarantee the exponential decay of $F(z_0, \xi)$ for z_0 near 0 in C^m.

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